Riesz transform for Dunkl Hermite expansion

Béchir Amri

University of Tunis, Preparatory Institut of Engineer Studies of Tunis, Department of Mathematics, 1089 Montfleury Tunis, Tunisia
bechir.amri@ipeit.rnu.tn

Abstract

In the present paper, we establish that Riesz transforms for Dunkl Hermite expansion as introduced in [4] are singular integral operators with Hörmander's type conditions and we show that are bounded on $L^p(\mathbb{R}^d, d\mu_\kappa)$, $1 < p < \infty$.

1 Introduction.

In [4] the authors introduced the Riesz transforms related to the Dunkl harmonic oscillator L_κ and they proved that when the group of reflections is isomorphic to $\mathbb{Z}_2^d$ such operators are L^p bounded with $1 < p < \infty$. The aim of this paper is to present an extension of this result to general group of reflections in arbitrary dimensions. Our approach consists in the application of the standard theory of Calderón-Zygmund operators. The setting, which is described in more details in section 2, is as follows: Let R be a (reduced) root system on $\mathbb{R}^d$ and G the associated reflection group. Set $\kappa : R \to [0, +\infty[$ a nonnegative multiplicity function on $\mathbb{R}^d$ and the weighted measure,

$$d\mu_\kappa(x) = \prod_{\alpha\in R} |\langle \alpha, x\rangle|^{2\kappa(\alpha)}, \quad x \in \mathbb{R}^d$$

where $\langle\ .\ \rangle$ is the Euclidean scalar product on $\mathbb{R}^d$. The Dunkl operators T_j, $(j = 1, ..., d)$ on $\mathbb{R}^d$ associated with G and κ are given by

$$T_j(f)(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in R_+} \kappa(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha.x)}{\langle x, \alpha\rangle}$$

where σ_α denotes the reflection in the hyperplane orthogonal to α. The Dunkl harmonic oscillator is the operator $L_\kappa = -\Delta_\kappa + \|x\|^2$ where Δ_κ denotes the Dunkl Laplacian operator $\Delta_\kappa = \sum_{j=1}^{d} T_j^2$. In particular the operator L_κ can be written as $L_\kappa = \frac{1}{2}\sum_{j=1}^{d}(\delta_j\delta_j^* + \delta_j^*\delta_j)$, where $\delta_j = T_j + x_j$ and $\delta_j^* = -T_j + x_j$.

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The Riesz transforms related to the Dunkl harmonic oscillator are defined as a natural generalizations of the classical ones (see [10]) by

$$R_j = \delta_j L_\kappa^{-\frac{1}{2}}, \quad j = 1...d.$$

The L^2 bounded of these operators can be easily obtained from the Dunkl Hermite expansions. Closely related to the integral operators the key new ingredient leading to L^p bounded is the following:

Theorem 1.1. *Let S be a bounded operator on $L^2(\mathbb{R}^d, d\mu_\kappa)$ and K be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \big\{(x, g.x);\ x \in \mathbb{R}^d,\ g \in G\big\}$ such that*

$$S(f)(x) = \int_{\mathbb{R}^N} K(x,y) f(y) d\mu_\kappa(y), \tag{1.1}$$

for all compactly supported f in $L^2(\mathbb{R}^d, d\mu_\kappa)$ with supp$(f) \cap G.x = \emptyset$. If K satisfies the Hörmander's type conditions: there exists a positive constant C such that for all $y, y_0 \in \mathbb{R}^d$,

$$\int_{\min_{g\in G} |g.x-y|>2|y-y_0|} |K(x,y) - K(x,y_0)|\, d\mu_\kappa(x) \ \leq\ C, \tag{1.2}$$

$$\int_{\min_{g\in G} |g.x-y|>2|y-y_0|} |K(y,x) - K(y_0,x)|\, d\mu_\kappa(x) \ \leq\ C, \tag{1.3}$$

then S extends to a bounded operator on $L^p(\mathbb{R}^d, d\mu_\kappa)$ for $1 < p \leq \infty$.

We will show that Riesz transform R_j has an integral representation satisfying conditions of Theorem 1.1.

Theorem 1.2. *The Riesz transforms R_j are bounded on $L^p(\mathbb{R}^d, d\mu_\kappa)$, $1 < p < \infty$.*

2 Background and outline of the proofs.

We first collect some background materials for the harmonic analysis associated with Dunkl operators. For more details see references [2, 3, 5, 7, 8, 6, 11] and the literature cited there.

The Dunkl kernel E_κ, associated with G and κ, is defined on $\mathbb{R}^d \times \mathbb{C}^d$ by: for $y \in \mathbb{C}^d$, $E_\kappa(., y)$ is the unique solution of the system:

$$T_j f = y_j f, \quad f(0) = 1. \tag{2.1}$$

It is symmetric in its arguments and has an unique holomorphic extension on $\mathbb{C}^d \times \mathbb{C}^d$. The Dunkl kernel is connected with the exponential function by the Bochner-type representation

$$E_\kappa(x,y) = \int_{\mathbb{R}^d} e^{\langle \eta, y\rangle} d\nu_x(\eta) \tag{2.2}$$

where ν_x is a probability measure supported in the the convex hull $co(G.x)$.

The Dunkl transform is defined, for $f \in L^1(\mathbb{R}^d, d\mu_\kappa)$ by:

$$\mathcal{F}_\kappa(f)(\xi) = \frac{1}{c_\kappa}\int_{\mathbb{R}^N} f(x)\, E_\kappa(-i\,\xi, x) d\mu_\kappa(x), \quad c_\kappa = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}}\, d\mu_\kappa(x).$$

It plays the same role as the Fourier transform in classical Fourier analysis ($\kappa \equiv 0$) and enjoys similar properties.

On $L^2(\mathbb{R}^d, d\mu\kappa)$ the Dunkl translation operator $\tau_x,\ x \in \mathbb{R}^d$ is defined by

$$\mathcal{F}_\kappa(\tau_x(f))(y) = E_\kappa(ix, y)\mathcal{F}_\kappa(f)(y), \quad y \in \mathbb{R}^d. \tag{2.3}$$

When f is a continuous radial function in $L^2(\mathbb{R}^d, d\mu_\kappa)$ with $f(y) = \widetilde{f}(|y|)$, an explicit formula of $\tau_x(f)$ is given by

$$\tau_x(f)(y) = \int_{\mathbb{R}^N} \widetilde{f}\Big(\sqrt{|x|^2 + |y|^2 + 2 < y, \eta >}\Big) d\nu_x(\eta). \tag{2.4}$$

This formula was first found by M. Rösler [8] for Schwartz functions and extended later to continuous functions by F. Dai and H. Wang [1]. The Dunkl translation operator satisfies

(i) For $f \in L^2(\mathbb{R}^d, d\mu_\kappa(x)) \cap L^1(\mathbb{R}^d, d\mu_\kappa(x))$

$$\int_{\mathbb{R}^d} \tau_x(f)(y) d\mu_\kappa(y) = \int_{\mathbb{R}^d} f(y) d\mu_\kappa(y). \tag{2.5}$$

(ii) For Schwartz function f

$$T_j\tau_x(f) = \tau_x(T_j f) \tag{2.6}$$

Let $\mathcal{P} = \mathbb{C}[\mathbb{R}^d]$ denote the algebra of polynomial functions on $\mathbb{R}^d$. The following bilinear form on $\mathcal{P}$,

$$[p, q]_\kappa = p(T)(q)(0) = c_\kappa \int_{\mathbb{R}^d} e^{\frac{-\Delta_\kappa}{2}} p(x) e^{\frac{-\Delta_\kappa}{2}} q(x) e^{\frac{-|x|^2}{2}} d\mu_\kappa(x)$$

define a scalar product, with some normalization constant $c_\kappa > 0$. For a given orthonormal basis $\{\varphi_n, n \in \mathbb{N}^d\}$ of $\mathcal{P}$ with respect to $[.,.]_\kappa$ we define the Hermite polynomials H_n and the Hermite functions h_n on $\mathbb{R}^d$,

$$H_n(x) = 2^{|n|} e^{\frac{-\Delta_\kappa}{2}} \varphi_n(x), \text{ and } h_n(x) = 2^{-\frac{|n|}{2}} e^{\frac{-|x|^2}{2}} H_n(x), n \in \mathbb{N}^d,$$

see[5] for more details. The important basic properties of the H_n and h_n are

(i) H_n satisfies the Mehler-formula,

$$\sum_{n \in \mathbb{N}^d} \frac{H_n(x)H_n(y)}{2^{|n|}} r^{|n|} = \frac{1}{(1-r^2)^{\gamma+\frac{d}{2}}} e^{-\frac{r^2}{1-r^2}(|x|^2+|y|^2)} E_\kappa(\frac{2r}{1-r^2}x, y). \tag{2.7}$$

(ii) h_n, $n \in \mathbb{N}^d$ are eigenfunctions of the operator L_κ, with

$$L_\kappa(h_n) = (2|n| + 2\gamma + d)h_n.$$

(iii) The set $\{h_n, n \in \mathbb{N}^d\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d; d\mu_\kappa)$.

Let $\mathcal{D}$, denotes the subspace of all finite linear combinations of $\{h_n, n \in \mathbb{N}^d\}$. The Riesz transform associated with L_κ is given on $\mathcal{D}$ by

$$R_j(f) = \delta_j L_\kappa^{-\frac{1}{2}}(f) = \sum_{n\in\mathbb{N}^d} (2|n| + 2\gamma + d)^{-\frac{1}{2}} \langle f, h_n\rangle \delta_j h_n, \quad f \in \mathcal{D}.$$

$L_\kappa^{-\frac{1}{2}}$ is defined on $L^2(\mathbb{R}^d, d\mu_\kappa)$ by

$$L_\kappa^{-\frac{1}{2}}(f) = \sum_{n\in\mathbb{N}^d} (2|n| + 2\gamma + d)^{-\frac{1}{2}} \langle f, h_n\rangle h_n.$$

Proposition 2.1. *The Riesz transform R_j^k extends to bounded operator from $L^2(\mathbb{R}^d; d\mu_\kappa)$ into itself.*

Proof. A short calculation shows that

$$\langle \delta_j f, g\rangle = \langle f, \delta_j^* g\rangle; \quad f, g \in \mathcal{D}.$$

Then with the notation

$$R_j^*(f) = \delta_j^* L_\kappa^{-\frac{1}{2}}(f) = \sum_{n\in\mathbb{N}^d} (2|n| + 2\gamma + d)^{-\frac{1}{2}} \langle f, h_n\rangle \delta_j^* h_n, \quad f \in \mathcal{D},$$

we have,

$$\begin{aligned}
\|R_j f\|_{2,k}^2 &\leq \|R_j f\|_{2,k}^2 + \left\|R_j^* f\right\|_{2,k}^2 \\
&= \left\langle \delta_j^* \delta_j L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f)\right\rangle + \left\langle \delta_j \delta_j^* L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f)\right\rangle \\
&= \left\langle \sum_{j=1}^d (\delta_j^* \delta_j + \delta_j \delta_j^*) L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f)\right\rangle \\
&= 2\left\langle L_\kappa L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f)\right\rangle = 2\int_{\mathbb{R}^d} |f(y)|^2 d\mu_\kappa(y).
\end{aligned}$$

Since $\mathcal{D}$ is a dense subspace in $L^2(\mathbb{R}^d; d\mu_\kappa)$, then R_j is uniquely extended to a bounded operator on $L^2(\mathbb{R}^d; d\mu_\kappa(x))$. □

2.1 Proof of Theorem 1.1

Considering the homogenous space $(\mathbb{R}^d, d\mu_\kappa)$, the proof follows essentially the proof of singular integral theorem of Calderon-Zygmund, see for example [9]. It consists of showing that the operator S associated to the kernel K is of weak type (1,1) and we can therefore concluded Theorem 1.1 by interpolation and duality.

Let $f \in L^1(\mathbb{R}^d, d\mu_\kappa) \cap L^2(\mathbb{R}^d, d\mu_\kappa)$ and $\lambda > 0$, there exist a decomposition of f, $f = h + b$ with $b = \sum_j b_j$ and a sequence of balls $(B(y_j, r_j))_j = (B_j)_j$ such that for some constant C, depending only on the multiplicity function κ,

(i) $\|h\|_\infty \le C\lambda$;

(ii) $supp(b_j) \subset B_j$;

(iii) $\displaystyle\int_{B_j} b_j(x)d\mu_\kappa(x) = 0$;

(iv) $\int_{\mathbb{R}^d} |b_j(y)|d\mu_\kappa(y) \le C\,\lambda\,\mu_\kappa(B_j)$;

(v) $\displaystyle\sum_j \mu_\kappa(B_j) \le \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y)$.

We will show that the following inequality hold for $w = h$ and $w = b$:

$$\rho_\lambda(S(w)) = \mu_\kappa\Big(\{x \in \mathbb{R}^d;\ |S(w)(x)| > \frac{\lambda}{2}\}\Big) \le \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y)dy. \tag{2.8}$$

The L^2-boundedness of S implies that

$$\begin{aligned}\rho_\lambda(S(h)) &\le \frac{4}{\lambda^2}\int_{\mathbb{R}^d} |S(h)(x)|^2 d\mu_\kappa(x)\\ &\le \frac{C}{\lambda^2}\int_{\mathbb{R}^d} |h(x)|^2 d\mu_\kappa(x)\\ &\le \frac{C}{\lambda^2}\int_{\bigcup_j B_j} |h(x)|^2 d\mu_\kappa(x) + \frac{C}{\lambda^2}\int_{\cup_j B_j)^c} |h(x)|^2 d\mu_\kappa(x).\end{aligned}$$

In view of (i) and (v),

$$\int_{\cup B_j} |h(x)|^2 d\mu_\kappa(x) \le C\lambda^2\mu_\kappa(\cup B_j) \le C\lambda\int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y)$$

and since $f(x) = g(x)$ on $x \in (\bigcup_j B_j)^c$,

$$\int_{(\cup_j B_j)^c} |h(x)|^2 d\mu_\kappa(x) \le C\lambda\int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y).$$

The inequality (2.8) is then satisfied for h.

In order to prove the inequality (2.8) for the function b, we put

$$B_j^* = B(y_j, 2r_j\ ); \quad \text{and} \quad Q_j^* = \bigcup_{g\in G} g.B_j^*$$

and we have

$$\rho_\lambda(S(b)) \le \mu_\kappa\Big(\bigcup_j Q_j^*\Big) + \mu_\kappa\left\{x \in \Big(\bigcup_j Q_j^*\Big)^c; |S(b)(x)| > \frac{\lambda}{2}\right\}.$$

Using the volume doubling property of the measure μ_κ with *(v)*,

$$\mu_\kappa\Big(\bigcup_j Q_j^*\Big) \le |G|\sum_j \mu_\kappa(B_j^*) \le C\sum_j \mu_\kappa(B_j) \le \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y).$$

Furthermore if $x \notin Q_j^*$, we have

$$\min_{g\in G} |g.x - y_j| > 2|y - y_j|, \quad y \in B_j.$$

Thus, from (1.1),*(iii)* ,*(ii)*, (1.2), *(iv)* and *(v)*

$$\begin{aligned}
&\int_{(\cup Q_j^*)^c} |S(b)(x)|d\mu_\kappa(x) \\
&\qquad \leq \sum_j \int_{(Q_j^*)^c} |S(b_j)(x)|d\mu_\kappa(x) \\
&\qquad = \sum_j \int_{(Q_j^*)^c} \left| \int_{\mathbb{R}^N} K(x,y)b_j(y)d\mu_\kappa(y)\right| d\mu_\kappa(x) \\
&\qquad = \sum_j \int_{(Q_j^*)^c} \left| \int_{\mathbb{R}^N} b_j(y)\Big(K(x,y) - K(x,y_j)\Big)d\mu_\kappa(y)\right| d\mu_\kappa(x) \\
&\qquad \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{(Q_j^*)^c} |K(x,y) - K(x,y_j)|\, d\mu_\kappa(x)d\mu_\kappa(y) \\
&\qquad \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{\min_{g\in G}|g.x-y_j|>2|y-y_j|} |K(x,y) - K(x,y_j)| d\mu_\kappa(x)d\mu_\kappa(y) \\
&\qquad \leq C\sum_j \int_{\mathbb{R}^d} |b_j(y)|d\mu_\kappa(y) \leq C \int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y).
\end{aligned}$$

We immediately get

$$\mu_\kappa \left\{ x \in \Big(\bigcup_j Q_j^*\Big)^c ; |S(b)(x)| > \frac{\lambda}{2} \right\} \leq \frac{2}{\lambda} \int_{(\cup Q_j^*)^c} |S(b)(x)|d\mu_\kappa(x) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(y)|d\mu_\kappa(y).$$

and (2.8) for b follows.

2.2 Proof of Theorem 1.2

To prove Theorem 1.2 we need several lemmas. Let us start by writing the Riesz transform as an integral operator given by (1.1).

The Hermite semigroup e^{-tL_κ} ($t \geq 0$), is given on $L^2(\mathbb{R}^d;\mu_\kappa)$ by

$$e^{-tL_\kappa}(f) = \sum_{n\in\mathbb{N}^d} e^{-t(2|n|+2\gamma+d)} \langle f, h_n\rangle h_n, \quad f \in L^2(\mathbb{R}^d;\mu_\kappa)$$

and has the following integral representation

$$e^{-tL_\kappa}(f)(x) = \int_{\mathbb{R}^d} k_t(x,y)f(y)d\mu_\kappa(y) \tag{2.9}$$

where

$$k_t(x,y) = \sum_{n\in\mathbb{N}^d} e^{-t(2|n|+2\gamma+d)} h_n(x)h_n(y).$$

The Mehler-formula (2.7) for the H_n leads to

$$\begin{aligned} k_t(x,y) &= e^{-\frac{1}{2}(|x|^2+|y|^2)}e^{-t(2\gamma+d)}\sum_{n\in\mathbb{N}^d} e^{-2t|n|}\frac{H_n(x)H_n(y)}{2^{|n|}} \\ &= \frac{1}{(sinh2t)^{\gamma+\frac{d}{2}}}e^{-\frac{(coth2t}{2}(|x|^2+|y|^2)}E_\kappa(\frac{x}{sinh2t},y). \end{aligned}$$

Moreover, by (2.2)

$$\begin{aligned} k_t(x,y) &= \frac{1}{(sinh2t)^{\gamma+\frac{d}{2}}}e^{-\frac{coth2t}{2}(|x|^2+|y|^2)}\int_{\mathbb{R}^d} e^{\frac{1}{sinh2t}\langle y,\eta\rangle}d\nu_x(\eta) \\ &= \frac{1}{(sinh2t)^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{coth2t}{2}(|x|^2+|y|^2-2\langle y,\eta\rangle)}e^{-tht\langle y,\eta\rangle}d\nu_x(\eta) \\ &= \frac{1}{(sinh2t)^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{coth2t}{2}|y-\eta|^2-tht\langle y,\eta\rangle}e^{-\frac{coth2t}{2}(|x|^2-|\eta|^2)}d\nu_x(\eta). \end{aligned}$$

Let k_t^0 the kernel of the classical Hermite semigroup (see [10]), given by

$$k_t^0(x,y) = \frac{1}{(sinh2t)^{\frac{d}{2}}}e^{-\frac{coth(2t)}{2}|x-y|^2-tht\langle x,y\rangle}.$$

Thus we have

$$k_t(x,y) = \frac{1}{(sinh2t)^{\gamma}}\int_{\mathbb{R}^d} k_t^0(\eta,y)e^{-\frac{coth(2t)}{2}(|x|^2-|\eta|^2)}d\mu_y(\eta). \tag{2.10}$$

As in [10] we shall use some estimates for the kernel k_t. From (2.10) this can easily done by investigating estimates of [10] for the kernel k_t^0 listed below.

Lemma 2.2 ([10]). *For $0 < t < 1$, there exist positive constants C and a independent of x, y and t, such that*

(i) $\left|k_t^0(x,y)\right| \le Ct^{-\frac{d}{2}}e^{\frac{-a|x-y|^2}{t}}$.

(ii) $\left|y_jk_t^0(x,y)\right| \le Ct^{-\frac{d+1}{2}}e^{\frac{-a|x-y|^2}{t}}$.

(iii) $\left|\frac{\partial k_t^0}{\partial y_j}(x,y)\right| \le Ct^{-\frac{d+1}{2}}e^{\frac{-a|x-y|^2}{t}}$.

(iv) $\left|y_j\frac{\partial k_t^0}{\partial y_j}(x,y)\right| \le Ct^{-\frac{d}{2}-1}e^{\frac{-a|x-y|^2}{t}}$.

Lemma 2.3 ([10]). *For $t \ge 1$, there exist positive constants C and a independent of x, y and t such that,*

(i) $\left|k_t^0(x,y)\right| \le Ce^{-dt}e^{-a|x-y|^2}$.

(ii) $\left|y_jk_t^0(x,y)\right| \le Ce^{-dt}e^{-a|x-y|^2}$.

Consequently we obtain the following:

Lemma 2.4. *For $0 < t < 1$, there exist positive constants C and b independent of x, y and t such that,*

(i) $\Big|k_t(x,y)\Big| \le Ct^{-\gamma-\frac{d}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)$.

(ii) $\Big|y_j k_t(x,y)\Big| \le Ct^{-\gamma-\frac{d+1}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)$.

(iii) $\Big|\dfrac{\partial k_t}{\partial y_j}(x,y)\Big| \le Ct^{-\gamma-\frac{d+1}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)$.

(iv) $\Big|y_j\dfrac{\partial k_t}{\partial y_j}(x,y)\Big| \le Ct^{-\gamma-\frac{d}{2}-1}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)$.

Proof. In view of (2.10), Lemma 2.2 and (2.4)

$$\begin{aligned}
\Big|k_t(x,y)\Big| &\le \frac{1}{(sinh2t)^\gamma}\int_{\mathbb{R}^d}\Big|k_t^0(\eta,y)\Big|e^{-\frac{coth(2t)}{2}(|x|^2-|\eta|^2)}d\nu_x(\eta)\\
&\le Ct^{-\gamma-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\frac{-a|y-\eta|^2}{t}}e^{-\frac{1}{4t}(|x|^2-|\eta|^2)}d\nu_x(\eta)\\
&\le Ct^{-\gamma-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-b\,\frac{|\eta-y|^2+|x|^2-|\eta|^2}{t}}d\nu_x(\eta) = Ct^{-\gamma-\frac{d}{2}}\tau_x(e^{\frac{-b}{t}|.|^2})(-y).
\end{aligned}$$

which prove (i) with $b = min(a, \frac{1}{4})$. We obtain (ii), (iii) and (iv) by similar way. □

Lemma 2.5. *For $t \ge 1$, there exist positive constants C and b independent of x, y and t such that,*

(i) $\Big|k_t(x,y)\Big| \le Ce^{-(2\gamma+d)t}\tau_x(e^{-b|.|^2})(-y)$

(ii) $\Big|y_j k_t(x,y)\Big| \le Ce^{-(2\gamma+d)t}\tau_x(e^{-b|.|^2})(-y)$

The proof is very similar to that of the previous lemma.

Lemma 2.6. *For $0 < t < 1$, there exist positive constants C and b independent of x, y and t such that, for $0 < t < 1$ and $1 \le i, j \le d$,*

(i) $\Big|(x_j - y_j)k_t(x,y)\Big| \le Ct^{-\gamma-\frac{d}{2}+\frac{1}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)$.

(ii) $\Big|(x_j - y_j)\dfrac{\partial k_t}{\partial y_i}(x,y)\Big| \le Ct^{-\gamma-\frac{d}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)$.

Proof. In view of (2.4) and (2.2) we have that

$$\tau_x(e^{\frac{-b}{t}|.|^2})(-y) = e^{\frac{-b}{t}(|x|^2+|y|^2)}E_\kappa(\frac{2b}{t}x, y).$$

From which and (2.1)

$$T_j\tau_x(e^{\frac{-b}{t}|.|^2})(-y) = \frac{2b}{t}(y_j - x_j)\tau_x(e^{\frac{-b}{t}|.|^2})(-y).$$

So, using (2.6) we get

$$\begin{aligned}
\Big|(y_j - x_j)k_t(x,y)\Big| &\leq Ct^{-\gamma-\frac{d}{2}}\Big|(x_j - y_j)\tau_x(e^{\frac{-b}{t}|.|^2})(-y)\Big| \\
&= Ct^{-\gamma-\frac{d}{2}}\Big|\frac{t}{2b}T_j\tau_x(e^{\frac{-b}{t}|.|^2})(-y)\Big| \\
&= Ct^{-\gamma-\frac{d}{2}}\Big|\frac{t}{2b}\tau_x(T_je^{\frac{-b}{t}|.|^2})(-y)\Big| \\
&\leq Ct^{-\gamma-\frac{d}{2}}\ \tau_x(|.|e^{\frac{-b}{t}|.|^2})(-y) \\
&\leq Ct^{-\gamma-\frac{d}{2}+\frac{1}{2}}\ \tau_x(e^{\frac{-b}{2t}|.|^2})(-y)\ .
\end{aligned}$$

It's the same for (ii). $\square$

Next, we put

$$\begin{aligned}
K_j(x,y) &= \frac{1}{\sqrt{\pi}}\int_0^{+\infty}\delta_j k_t(x,y)\frac{dt}{\sqrt{t}} \\
&= \frac{1}{\sqrt{\pi}}\int_0^{+\infty}k_t(x,y)\Big((1-coth2t)x_j + \frac{1}{sinh2t}y_j\Big)\frac{dt}{\sqrt{t}}. \qquad (2.11)
\end{aligned}$$

Lemma 2.7. *For all $x, y \in \mathbb{R}^d$, $y \notin G.x$, the integral (2.11) converge absolutely and*

$$|K_j(x,y)| \leq C\ \frac{1}{\min_{g\in G}|y - g.x|^{2\gamma+d}}$$

Proof. Let us first noting that for $x, y, \eta \in \mathbb{R}^d$, $\eta \in co(G.x)$

$$\max_{g\in G}|y - g.x| \leq \sqrt{|x|^2 + |y|^2 + 2 < y,\eta >} \leq \max_{g\in G}|y - g.x|. \qquad (2.12)$$

In view of Lemma 2.4, Lemma 2.6, (2.4) and (2.12)

$$\begin{aligned}
&\int_0^1\Big|k_t(x,y)\Big((1-coth2t)x_j + \frac{1}{sinh2t}y_j\Big)\Big|\frac{dt}{\sqrt{t}} \\
&= \int_0^1\Big|k_t(x,y)\Big((1-coth2t)(x_j - y_j) + (1-\tanh t)y_j\Big)\Big|\frac{dt}{\sqrt{t}} \\
&\leq C\int_0^1 t^{-\gamma-\frac{d+1}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)\frac{dt}{\sqrt{t}} \\
&= C\int_{\mathbb{R}^d}\int_0^1 t^{-\gamma-\frac{d+1}{2}}e^{\frac{-b}{t}(|x|^2+|y|^2-2\langle y,\eta\rangle)}d\nu_x(\eta)\frac{dt}{\sqrt{t}} \\
&= C\int_{\mathbb{R}^d}\frac{1}{(|x|^2+|y|^2-2\langle y,\eta\rangle)^{\gamma+\frac{d}{2}}}\Big(\int_0^{|x|^2+|y|^2-2\langle y,\eta\rangle} u^{-\gamma-\frac{d+2}{2}}\ e^{\frac{-b}{u}}du\Big)d\nu_x(\eta) \\
&\leq C\ \frac{1}{\min_{g\in G}|y - g.x|^{2\gamma+d}}\ .
\end{aligned}$$

However, from Lemma 2.5 it follows that

$$\int_1^{+\infty}\Big|k_t(x,y)\Big((1-coth2t)x_j+\frac{1}{sinh2t}y_j\Big)\Big|\frac{dt}{\sqrt{t}}$$

$$\leq C\tau_x(e^{-b|.|^2})(-y)\leq Ce^{-b\min_{g\in G}|y-g.x|^2}\leq C\ \frac{1}{\min_{g\in G}|y-g.x|^{2\gamma+d}}\ .$$

which proves the result. □

Proposition 2.8. *The Riesz transform R_j^k satisfies*

$$R_j^k(f)(x)=\int_{\mathbb{R}^d}K_j(x,y)f(y)d\mu_\kappa(y)$$

for all compactly supported function $f\in L^2(\mathbb{R}^d;d\mu_\kappa)$ with $G.x\cap supp(f)=\emptyset$.

Proof. We first write $L_\kappa^{-\frac{1}{2}}$ in the folowing way

$$L_\kappa^{-\frac{1}{2}}(f)(x)\ =\ \frac{1}{\sqrt{\pi}}\int_0^{+\infty}e^{-tL_\kappa}(f)(x)\frac{dt}{\sqrt{t}}.$$

Therefore, (2.9) and Fubin's Theorem yield

$$L_\kappa^{-\frac{1}{2}}(f)(x)=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^d}\int_0^{+\infty}k_t(x,y)f(y)d\mu_\kappa(y)\frac{dt}{\sqrt{t}}.$$

Following the proof of Lemma 2.7, the last integral converge absolutely when $G.x\cap supp(f)=\emptyset$. Indeed, by Lemma 2.5

$$\int_{\mathbb{R}^d}\int_0^1\Big|k_t(x,y)f(y)\Big|d\mu_\kappa(y)\frac{dt}{\sqrt{t}}$$

$$\begin{aligned}
&\leq\ C\int_{\mathbb{R}^d}\int_0^1 t^{-\gamma-\frac{d}{2}}\ \tau_x(e^{\frac{-b}{t}|.|^2})(-y)|f(y)|\frac{dt}{\sqrt{t}}d\mu_\kappa(y)\\
&=\ C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 t^{-\gamma-\frac{d}{2}}e^{\frac{-b}{t}(|x|^2+|y|^2-2\langle y,\eta\rangle)}|f(y)|\frac{dt}{\sqrt{t}}d\nu_x(\eta)d\mu_\kappa(y)\\
&\leq\ C\int_{\mathbb{R}^d}\frac{|f(y)|}{\min_{g\in G}|y-g.x|^{2\gamma+d-1}}d\mu_\kappa(y).
\end{aligned}$$

In addition, applying Lemma 2.7 we get

$$\int_{\mathbb{R}^d}\int_1^{+\infty}\Big|k_t(x,y)f(y)\Big|d\mu_\kappa(y)\frac{dt}{\sqrt{t}}$$

$$\begin{aligned}
&\leq\ C\int_{\mathbb{R}^d}\int_1^{+\infty}e^{(-2\gamma-d)t}\ \tau_x(e^{-b|.|^2})(-y)|f(y)|\frac{dt}{\sqrt{t}}d\mu_\kappa(y)\\
&=\ C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_1^{+\infty}e^{(-2\gamma-d)t}e^{-b(|x|^2+|y|^2-2\langle y,\eta\rangle))}|f(y)|\frac{dt}{\sqrt{t}}d\nu_y(\eta)d\mu_\kappa(y)\\
&\leq\ \int_{\mathbb{R}^d}\int_1^{+\infty}e^{(-2\gamma-d)t}|f(y)|\frac{dt}{\sqrt{t}}d\nu_y(\eta)d\mu_\kappa(y)\\
&\leq\ C\int_{\mathbb{R}^d}|f(y)|d\mu_\kappa(y).
\end{aligned}$$

Now, making use of (ii) of Lemma 2.7 and (2.12), we deduce that

$$\Big|\frac{\partial k_t}{\partial x_j}(x,y)\Big| = \Big|\frac{\partial k_t}{\partial y_j}(y,x)\Big| \leq \begin{cases} t^{-\gamma-\frac{d}{2}}e^{-\frac{b}{t}\min_{g\in G}|g.x-y|^2}; & \text{if } 0<t<1. \\ e^{-(2\gamma+d)t}e^{-b\min_{g\in G}|g.x-y|^2}; & \text{if } t\geq 1. \end{cases}$$

Thus, by Leibnitz's Rule for differentiation of integrals

$$\begin{aligned} R_j^k(f)(x) &= \delta_j L_\kappa^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^d}\int_0^{+\infty}\delta_j k_t(x,y)f(y)\frac{dt}{\sqrt{t}}d\mu_\kappa(y) \\ &= \int_{\mathbb{R}^d} K_j(x,y)f(y)d\mu_\kappa(y). \end{aligned}$$

and the proof of Proposition 2.8 follows. □

Proposition 2.9. *There exists a positive constant C such that for all $y, y_0 \in \mathbb{R}^d$*

$$\int_{\min_{g\in G}|g.x-y|>2|y-y_0|}\Big|K_j(x,y)-K_j(x,y_0)\Big|d\mu_\kappa(y) \leq C \tag{2.13}$$

$$\int_{\min_{g\in G}|g.x-y|>2|y-y_0|}\Big|K_j(y,x)-K_j(y_0,x)\Big|d\mu_\kappa(y) \leq C \tag{2.14}$$

Proof. We will only show (2.13), since the proof of (2.14) follows in a similar way. Put,

$$h_t(x,y) = \delta_j k_t(x,y) = k_t(x,y)\Big((1-coth2t)x_j + \frac{1}{sinh2t}y_j\Big).$$

Then in view of (2.11), for all $x, y \in \mathbb{R}^d$, $y \notin G.x$

$$\begin{aligned} K_j(x,y) &= \frac{1}{\sqrt{\pi}}\int_0^1 h_t(x,y)\frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}}\int_1^{+\infty} h_t(x,y)\frac{dt}{\sqrt{t}}. \\ &= K_j^{(1)}(x,y) + K_j^{(2)}(x,y). \end{aligned}$$

From Lemma 2.7 we have

$$\begin{aligned} \int_{\mathbb{R}^d}|K_j^{(2)}(x,y)|d\mu_\kappa(x) &\leq C\int_{\mathbb{R}^d}\int_1^{+\infty} e^{(-2\gamma-d)t}\tau_{-y}(e^{\frac{-a}{t}|.|^2})(x)|\frac{dt}{\sqrt{t}}d\mu_\kappa(x) \\ &\leq C\int_1^{+\infty}\int_{\mathbb{R}^d} e^{(-2\gamma-d)t}e^{\frac{-a}{t}|z|^2}d\mu_\kappa(z)\frac{dt}{\sqrt{t}} \leq C. \end{aligned}$$

Therefore,

$$\int_{\min_{g\in G}|g.x-y|>2|y-y_0|}\Big|K_j^{(2)}(x,y)-K_j^{(2)}(x,y_0)\Big|d\mu_\kappa(x) \leq 2\int_{\mathbb{R}^d}|K_2(x,y)d\mu_\kappa(x) \leq C.$$

However, the estimates of Lemmas 2.4 and 2.6 give

$$\Big|\frac{\partial h_t}{\partial y_i}(x,y)\Big| \leq Ct^{-\gamma-\frac{d}{2}-1}\,\tau_x(e^{\frac{-b}{t}|.|^2})(-y), \quad 0<t<1$$

where

$$\frac{\partial h_t}{\partial y_i}(x,y) = \begin{cases} \frac{\partial k_t}{\partial y_i}(x,y)\Big((1-\coth 2t)(x_j-y_j)+(1-\tanh t)y_j\Big), & \text{if } i\neq j \\ \frac{\partial k_t}{\partial y_j}(x,y)\Big((1-\coth 2t)(x_j-y_j)+(1-\tanh t)y_j\Big) + \frac{1}{\sinh 2t}k_t(x,y). \end{cases}.$$

By mean value theorem,

$$\begin{aligned}\left|K_j^{(1)}(x,y)-K_j^{(1)}(x,y_0)\right| &= \frac{1}{\sqrt{\pi}}\int_0^1 |h_t(x,y)-h_t(x,y_0)|\frac{dt}{\sqrt{t}}\\ &\le \frac{1}{\sqrt{\pi}}|y-y_0|\int_0^1\int_0^1\sum_{i=1}^d\left|\frac{\partial h_t}{\partial y_i}(x,y_\theta)\right|d\theta\frac{dt}{\sqrt{t}}\\ &\le C|y-y_0|\int_0^1\int_0^1 t^{-\gamma-\frac{d}{2}-1}\,\tau_x(e^{\frac{-b}{t}|.|^2})(-y_\theta)d\theta\frac{dt}{\sqrt{t}}\end{aligned}$$

where $y_\theta = y_0+\theta(y-y_0)$. If $\min_{g\in G}|g.x-y|>2|y-y_0|$, we have that

$$\min_{g\in G}|g.x-y_\theta|\ge\min_{g\in G}|g.x-y|-|y-y_\theta|>|y-y_0|$$

and by (2.4)-(2.12)

$$\tau_x(e^{\frac{-b}{t}|.|^2})(-y_\theta)\le\tau_x\left(e^{\frac{-b}{2t}(|.|+|y-y_0|)^2}\right)(-y_\theta).$$

Hence by using (2.5), we get that

$$\int_{\min_{g\in G}|g.x-y|>2|y-y_0|}\left|K_j^{(1)}(x,y)-K_j^{(1)}(x,y_0)\right|d\mu_\kappa(x)$$

$$\begin{aligned}&\le C|y-y_0|\int_0^1\int_0^1 t^{-\gamma-\frac{d}{2}-1}\int_{\mathbb{R}^d}\tau_{-y_\theta}\left(e^{\frac{-b}{2t}(|.|+|y-y_0|)^2}\right)(x)d\mu_\kappa(x)\frac{dt}{\sqrt{t}}d\theta\\ &= C|y-y_0|\int_0^1 t^{-\gamma-\frac{d}{2}-1}\int_{\mathbb{R}^d}e^{\frac{-b}{2t}(|z|+|y-y_0|)^2}d\mu_\kappa(z)\frac{dt}{\sqrt{t}}\\ &\le C|y-y_0|\int_0^{+\infty}r^{2\gamma+d-1}\int_0^1 t^{-\gamma-\frac{d}{2}-1}e^{\frac{-b}{2t}(r+|y-y_0|)^2}\frac{dt}{\sqrt{t}}\,dr\\ &\le C|y-y_0|\int_0^{+\infty}\frac{r^{2\gamma+d-1}}{(r+|y-y_0|)^{2\gamma+d+1}}\,dr\int_0^{+\infty}u^{-\gamma-\frac{d}{2}-1}e^{\frac{-b}{2u}}\frac{du}{\sqrt{u}}\\ &\le C|y-y_0|\int_0^{+\infty}\frac{1}{(r+|y-y_0|)^2}\,dr\le C.\end{aligned}$$

This finishes the proof of proposition 2.9 and concluded Theorem 1.2. □

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